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POLYNOMIAL FITTING WITH MEAN DERIVATIVES AND MEAN DIFFERENCES

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by

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ABSTRACT

The problem of developing orthogonal polynomials for finding a least-squares best approximation to a continuous function over a given interval is solved by a method of finding the mean k^{th} derivative of the function over that interval. A parallel method, using mean differences, is developed for fitting discrete data.

FOREWORD

This work was carried out as part of an independent research project entitled Applied Mathematics and Numerical Analysis as authorized by WEPTASK No. 000 000 000/107-1/R0110101. The primary reason for the study was the need for improved methods in the functionalization of parameters used in connection with trajectories and aiming data. Credit is due many of the present and past Naval Weapons Laboratory employees. The following contributions to the solution of the problem deserve special credit:

Mr. J. F. Parker initiated use of orthogonal polynomials on the first NWL digital computer; Mr. John H. Walker, Mr. Ira V. West, and Mr. Larry E. Spangler developed the technique reported in Appendix A; Mr. Bruce Bosley and Dr. Hartmut Huber prepared the tables in Appendix B. Mr. Bruce Bransom made a study of the method while a student at the University of South Dakota and Mrs. Jane H. Martina aided in checking and editing the report.

RELEASED BY:

23 Jan 1969

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Head, Warfare Analysis Department

CONTENTS

INTRODUCTION

This report is an outgrowth of the work reported in NWL Report No. 1822. Although much was done in this field long before NWL Report No. 1822 appeared, and outstanding contributions have been made by others during the preparation of this report, it is felt that the following will be of interest to those working in the field:

(1) The use of mean derivatives and mean differences to tie the discrete case to the continuous case and to develop corresponding weights for the discrete case.

(2) The simple method of fitting with certain type constraints.

(3) The use of difference matrices in the fitting of a function of two independent variables.

Appendix B gives matrices built from m values for making Jacobi-type fits to equally spaced discrete data. For higher degree and more points it is expected that a high-speed computer would compute its own m values. A supplementary report is planned for giving the programs to be used in computer evaluation of w and m values, as well as a more extensive tabulation of the w's and m's which would be convenient for use with the desk calculator.

MEAN VALUES OF DERIVATIVES OVER A CONTINUOUS SET

Suppose that the approximation

$$f = A_0P_0 + A_1P_1 + A_2P_2 + \dots A_nP_n , \tag{1}$$

where $P_k = P_k(x)$, a kth degree polynomial, $a \le x \le b$, is desired. If equation (1) is differentiated, the following set of equations is obtained:

$$\begin{aligned} f &= A_0P_0 + A_1P_1 + A_2P_2 + \dots + A_nP_n \\ f' &= A_1P_1' + A_2P_2' + \dots + A_nP_n' \\ f'' &= A_2P_2'' + \dots + A_nP_n'' \\ &\vdots \\ f^{(n)} &= A_nP_n^{(n)} \end{aligned} \tag{2}$$

The matrix of the coefficients of the P's is triangular and the system of equations (2) can be used to solve for the A's in terms of f and P values.

If x_1, $a \le x_1 \le b$, is substituted in equations (2) and the system solved, a truncated Taylor expansion about $x = x_1$ is obtained. The Taylor expansion is, of course, in terms of derivatives evaluated at a point.

Let it be supposed that instead of using the k^{th} derivative at $x = x_1$, a mean of the k^{th} derivative over the set $a \le x \le b$ is to be used. If w is an integrable function, not changing signs over the set $a \le x \le b$, normalized so that

$$\int_a^b w dx = 1,$$

the mean k^{th} derivative of f is (omitting the limits of integration, a and b)

$$\overline{f^{(k)}} = \int w f^{(k)} dx . \qquad (3)$$

If additional restrictions

$$\begin{aligned} w(a) &= w'(a) = \ldots w^{(k-1)}(a) = 0 \\ w(b) &= w'(b) = \ldots w^{(k-1)}(b) = 0 \end{aligned} , \qquad (4)$$

are made, then equation (3) can be changed, integrating by parts, to a form which might be easier for evaluation:

$$
\begin{aligned}
\int w f^{(k)} dx &= -\int w' f^{(k-1)} dx \\
&= \int w'' f^{(k-2)} dx \\
&\ \ \vdots \\
&= (-1)^k \int w^{(k)} f dx \\
&= \overline{f^{(k)}}
\end{aligned}
\tag{5}
$$

If $w^{(k)}$ is written as $w^{(k)} = \rho Q_k$, with $\rho \geq 0$ for $a \leq x \leq b$,

$\int \rho dx = L$, a finite number,

Q_k = a k^{th} degree polynomial in x, the Q polynomials are orthogonal, with weight ρ, with respect to integration over the set $a \leq x \leq b$.

This is shown by

$$
\begin{aligned}
\int \rho Q_i Q_j dx &= \int \rho Q_j Q_i dx \qquad (i < j) \\
&= \overline{Q_i^{(j)}} = 0
\end{aligned}
\tag{6}
$$

and

$$
\int \rho Q_i Q_i dx = \int \rho Q_i^2 \, dx > 0. \tag{7}
$$

Special Cases

I. Legendre Polynomials

Let $a = -1$ and $b = 1$, c = a constant factor, and take

$$cw = (1 - x^2)^k ,$$
$$cw' = k(-2x)\ (1-x^2)^{k-1} ,$$
$$cw'' = k(-2)\ (1-x^2)^{k-1} + k(k-1)\ (-2x)^2\ (1-x^2)^{k-2} ,$$
$$\vdots$$

and $\quad w^{(k)} \doteq Q_k$, a polynomial of k^{th} degree.

With the proper choice of c, Q_k is the k^{th} Legendre polynomial. This is the case for which ρ is constant and Legendre polynomials are integration-wise orthogonal over the continuous set, $-1 \le x \le 1$, with respect to a constant weighting function.

II. Chebyshev Polynomials

Let a = -1 and b = 1, and take

$$cw = (1-x^2)^{k-1/2} ,$$
$$cw' = (k-1/2)\ (-2x)\ (1-x^2)^{k-1-1/2} ,$$
$$cw'' = (k-1/2)\ (k-1-1/2)\ (-2x)^2\ (1-x^2)^{k-2-1/2} ,$$
$$+ (k-1/2)\ (-2)\ (1-x^2)\ (1-x^2)^{k-2-1/2} ,$$
$$= Q_2(x)\ (1-x^2)^{k-2-1/2} ,$$
$$\vdots$$

and $\quad w^{(k)} \doteq Q_k (1-x^2)^{-1/2}$.

With the proper choice of c, Q_k is the k^{th} Chebyshev polynomial. The weighting function is a constant times $(1-x^2)^{-1/2}$.

III. Jacobi Polynomials

For $a = -1$, $b = 1$:

$$cw = (1-x)^{\alpha+k} (1+x)^{\beta+k} , \alpha > -1, \beta > -1,$$

$$cw' = (\alpha+k)(-1)(1+x)(1-x)^{\alpha+k-1} (1+x)^{\beta+k-1} ,$$

$$+ (\beta+k)(1)(1-x)(1-x)^{\alpha+k-1} (1+x)^{\beta+k-1} ,$$

$$= Q_1(1-x)^{\alpha+k-1} (1+x)^{\beta+k-1} ,$$

$$cw'' = Q_1'(1-x)(1+x)(1-x)^{\alpha+k-2} (1+x)^{\beta+k-2} ,$$

$$+ Q_1(\alpha+k-1)(-1)(1+x)(1-x)^{\alpha+k-2} (1-x)^{\beta+k-2} ,$$

$$+ Q_1(\beta+k-1)(1)(1-x)(1-x)^{\alpha+k-2} (1+x)^{\beta+k-2} ,$$

$$= Q_2(1-x)^{\alpha+k-2} (1+x)^{\beta+k-2} ,$$

$$\vdots$$

and $$w^{(k)} = Q_k(1-x)^{\alpha} (1+x)^{\beta} .$$

With the proper choice of c, Q_k is the k^{th} Jacobi polynomial. The weighting function is a constant times $(1-x)^{\alpha} (1+x)^{\beta}$.

Note that for $\alpha = \beta = 0$, the Jacobi polynomials become the Legendre polynomials; for $\alpha = \beta = -1/2$, they become the Chebyshev polynomials.

The Legendre, Chebyshev, and Jacobi Polynomials are orthogonal over the interval $-1 \leq x \leq 1$. To find polynomials orthogonal over the arbitrary finite interval, $a \leq x \leq b$, a transformation in the independent variable can be made to bring the problem to the forms already studied; or, the Jacobi form for w can be taken as

$$cw = (x-a)^{\beta+k} (b-x)^{\alpha+k} .$$

IV. Hermite Polynomials

For the infinite interval $-\infty < x < \infty$, with c a constant factor, one can take

$$\begin{aligned} cw &= \exp(-\alpha^2 x^2) \\ cw' &= -2\alpha^2 x \exp(-\alpha^2 x^2) \\ cw'' &= (-2\alpha^2 + (-2\alpha^2 x)^2) \exp(-\alpha^2 x^2) \\ & \text{- - - - - - - - - - - - - - -} \end{aligned}$$

and $$w^{(k)} = Q_k \exp(-\alpha^2 x^2) .$$

Q_k is, with the proper choice of c and α, the k^{th} Hermite polynomial, the weighting function being a constant times $\exp(-\alpha^2 x^2)$.

V. Laguerre Polynomials

For the semi-infinite interval $0 \leq x < \infty$ one can take

$$cw = x^k \exp(-\alpha x)$$

$$cw' = (k-\alpha x)x^{k-1} \exp(-\alpha x)$$

$$cw'' = [-\alpha x + (k-\alpha x)(k-1-\alpha x)]x^{k-1} \exp(-\alpha x)$$

$$= Q_2 x^{k-2} \exp(-\alpha x)$$

$$- - - - - - - - - - - - - - - - - - - -$$

and $$w^{(k)} = Q_k \exp(-\alpha x) \ .$$

With the proper choice of c and α, Q_k is the k^{th} Laguerre polynomial; the weighting function is a constant times $\exp(-\alpha x)$.

The literature gives a more complete discussion of orthogonal polynomials and orthogonal functions (see reference (1)). It should be pointed out, that while use of orthogonal polynomials for the P's in equation (1) leads to a diagonal matrix of coefficients of the A's in the set of linear equations obtained by taking mean k^{th} derivatives, any desired form for the P's can be used and the resulting matrix of coefficients of the A's will be no worse than triangular. Further, any general non-negative w function can be used in equation (3) for finding the mean k^{th} derivative, and the resulting matrix of the coefficients of the A's will be no worse than triangular.

LEAST SQUARES FITS IN TERMS OF JACOBI POLYNOMIALS

Continuous Data

It will be assumed that the fit is over a finite interval $a \le x \le b$, and that a linear transformation has been made in the

argument, $u = L(x)$, in such a way that with the function expressed as $f(u)$ the fit is made over the interval $-1 \leq u \leq 1$.

The orthogonal polynomials have the property that

$$\int_{-1}^{1} \rho(u) Q_i(u) Q_j(u)\, du = 0, \qquad i \neq j, \tag{8}$$

with $\rho(u)$ being the weighting function.

The weighted least-squares fit

$$f(u) = A_o Q_o(u) + A_1 Q_1(u) + A_2 Q_2(u) + \ldots + A_n Q_n(u) \tag{9}$$

is made by defining

$$e(u) = f(u) - A_o Q_o(u) - A_1 Q_1(u) - A_2 Q_2(u) - \ldots - A_n Q_n(u)$$

and minimizing

$$E = \int_{-1}^{1} \rho(u) e^2(u)\, du \ .$$

Setting $\frac{\partial E}{\partial A_i} = 0$ leads to

$$
\left.
\begin{aligned}
\int_{-1}^{1} \rho(u)Q_o(u)f(u)du &= A_o \int_{-1}^{1} \rho(u)Q_o^{\,2}(u)du \\
&\quad + A_1 \int_{-1}^{1} \rho(u)Q_o(u)Q_1(u)du \\
&\quad\quad \vdots \\
&\quad + A_n \int_{-1}^{1} \rho(u)Q_o(u)Q_n(u)du \;, \\
\int_{-1}^{1} \rho(u)Q_1(u)f(u)du &= A_o \int_{-1}^{1} \rho(u)Q_o(u)Q_1(u)du \\
&\quad + A_1 \int_{-1}^{1} \rho(u)Q_1^{\,2}(u)du \\
&\quad\quad \vdots \\
&\quad + A_n \int_{-1}^{1} \rho(u)Q_1(u)Q_n(u)du \\
&\text{- -} \\
\int_{-1}^{1} \rho(u)Q_n(u)f(u)du &= A_o \int_{-1}^{1} \rho(u)Q_o(u)Q_n(u)du \\
&\quad\quad \vdots \\
&\quad + A_n \int_{-1}^{1} \rho(u)Q_n^{\,2}(u)du
\end{aligned}
\right\} \qquad (10)
$$

Because of the orthogonality property, the matrix of coefficients of the A's is diagonal, and for each k,

$$
\int_{-1}^{1} \rho(u)Q_k(u)f(u)du = A_k \int_{-1}^{1} \rho(u)Q_k^{\,2}(u)du \qquad (11)
$$

Discrete Data

Suppose f(x) is given for a discrete set of values of x, $a = x_0 < x_1 < x_2 \quad \ldots < x_m = b$, and that the transformation

$$u = \frac{2x-a-b}{b-a} \tag{12}$$

has been made so that the corresponding arguments, in u, are

$$u_0 = -1 < u_1 < u_2 < \cdots < u_m = 1.$$

Replace the integrals in equations (10) by

$$\int_{-1}^{1} \rho(u) Q_k(u) f(u) du \simeq$$

$$(u_1 - u_0) \frac{I(0,1)}{(u_1 - u_0)} \left[Q_k(u_0) f(u_0) + Q_k(u_1) f(u_1) \right]$$

$$+ (u_2 - u_1) \frac{I(1,2)}{(u_2 - u_1)} \left[Q_k(u_1) f(u_1) + Q_k(u_2) f(u_2) \right]$$

$$+ \cdots\cdots\cdots$$

$$+ (u_m - u_{m-1}) \frac{I(m-1,m)}{(u_m - u_{m-1})} \left[Q_k(u_{m-1}) f(u_{m-1}) + Q_k(u_m) f(u_m) \right] , \tag{13}$$

$$\int_{-1}^{1} \rho Q_j(u) Q_k(u) du \simeq$$

$$(u_1 - u_0) \frac{I(0,1)}{(u_1 - u_0)} \left[Q_j(u_0) Q_k(u_0) + Q_j(u_1) Q_k(u_1) \right]$$

$$+ \; . \; . \; . \; . \; . \; . \; .$$

$$+ (u_m - u_{m-1}) \frac{I(m-1,m)}{(u_m - u_{m-1})} \left[Q_j(u_{m-1}) Q_k(u_{m-1}) + Q_j(u_m) Q_k(u_m) \right] \quad (14)$$

where

$$I(r,s) = \frac{1}{2} \int_{u_r}^{u_s} \rho(u) du \quad . \quad (15)$$

Note that this is the result of replacing $\rho(u)$ in each interval $u_r < u < u_s$ by its mean in that interval and then integrating by the trapezoidal rule.

In matrix language, let the set of equations obtained by substituting $u = u_0, u_1, u_2, \ldots, u_n$ in equation (9) be

$$Q \vec{A} = \vec{F} \quad , \quad (16)$$

where

$$\vec{A} = \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{bmatrix} \quad ; \quad \vec{F} = \begin{bmatrix} f(u_0) \\ f(u_1) \\ f(u_2) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f(u_m) \end{bmatrix} ,$$

and

$$Q = \begin{bmatrix} Q_0(u_0) & Q_1(u_0) & Q_2(u_0) & \cdots & Q_n(u_0) \\ Q_0(u_1) & Q_1(u_1) & Q_2(u_1) & \cdots & Q_n(u_1) \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ Q_0(u_m) & Q_1(u_m) & Q_2(u_m) & \cdots & Q_n(u_m) \end{bmatrix}$$

The approximations in equations (13) and (14) substituted in equations (10) give, in matrix language,

$$Q^*D\ Q\ \vec{A} = Q^*\ D\ \vec{F} , \tag{17}$$

superscript * indicating transpose and D being a diagonal "weighting" matrix

$$\begin{bmatrix} I(0,1) & 0 & 0 & \cdots & 0 \\ 0 & I(0,2) & 0 & \cdots & 0 \\ 0 & 0 & I(1,3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & I(m-1,m) \end{bmatrix}$$

<u>Special Cases</u>

A. Legendre Polynomials

In this case, $\rho(u) = 1$ and $I(r,s) = \frac{1}{2}(u_s - u_r)$

B. Chebyshev Polynomials

In this case, $\rho(u) = (1 - u^2)^{-\frac{1}{2}}$

and

$$I(r,s) = \frac{1}{2}\left[\text{arcsine } u_s - \text{arcsine } u_r\right]$$

Note that the matrix, $Q^* D Q$, in equation (17) will not be diagonal, but will be near enough diagonal in form to give a good inverse.

If a P-matrix is built in the same manner as the Q-matrix was, the least squares solution of $P \vec{A} = \vec{F}$ can be found by solving

$$Q^* \; D \; P \vec{A} = Q^* \; D \; \vec{F} \quad . \tag{18}$$

The proof of this can be constructed from the fact that every $P_K(u)$ is a linear combination of orthogonal polynomials of degree K, K-1, --- 1, 0. The Q^* D P matrix will be almost triangular.

Needed values of Legendre polynomials can be computed using

$$P_0(u) = 1$$

$$P_1(u) = u$$

$$P_2(u) = \frac{1}{2} \; (3u^2 - 1)$$

$$\vdots$$

$$P_{r+1}(u) = \frac{1}{r+1} \; \left[(2r+1)uP_r(u) - rP_{r-1}(u)\right]$$

Needed values of Chebyshev polynomials can be generated using

$$T_0(u) = 1$$

$$T_1(u) = u$$

$$T_2(u) = 2u^2 - 1$$

$$\vdots$$

$$T_{r+1}(u) = 2uT_r(u) - T_{r-1}(u) \quad .$$

LEAST SQUARES FITS TO DISCRETE DATA OBTAINED BY MEAN DIFFERENCES

Let the form desired for the fit to (approximation of) a function be

$$\begin{bmatrix} P_0(u) & P_1(u) & P_2(u) & \ldots & P_n(u) \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{bmatrix} = f(u) \qquad (1')$$

If f(u) is known for a discrete set of values of u (greater than (n+1) in number) and substitution of these u values are made in equation (1') the result can be written

$$U \vec{A} = \vec{F} \quad ,$$

with each row of the U matrix being the row matrix in equation (1') evaluated for one of the u values for which f(u) is known and the vectors $\vec{A}$ and $\vec{F}$ being defined as in equation (17).

Suppose that multipliers can be found such that if these multipliers be used as components of a vector, the dot product of this vector and the $\vec{F}$ vector is a constant times the mean k^{th} difference of f. A matrix M can then be built with the first row being the multipliers for finding the mean zeroth difference, the second row being the multipliers for finding the mean first difference, ..., and the (n+1)st row being the multipliers for finding the mean nth difference. Then premultiplying by M, the

the equation

$$M\,U\,\vec{A} = M\,\vec{F}$$

is obtained.

But (see reference (2))

$$M\,U = T$$

with T a nonsingular triangular matrix. Therefore,

$$\begin{aligned}\vec{A} &= (MU)^{-1}\,M\,\vec{F} \qquad (19)\\ &= T^{-1}\,M\,\vec{F}\end{aligned}$$

It should be noted that construction of the multipliers for unequal spacing is much less attractive than is the case for equal spacing, and, in general, is to be avoided unless the particular grid of the unequal spacing is to be used repeatedly.

Identification of fits thus obtained with least squares fits can be made by following a path parallel to that used in developing the Jacobi polynomials by finding mean k^{th} derivatives.

Let μ^{-1} be a positive definite, symmetric, weighting matrix and Q be a matrix whose (k+1)st column is built from evaluations, at a discrete set of arguments, of a k^{th} degree polynomial, Q_k, the Q_k's being constructed in such a way that

$$Q^{*}\,\mu^{-1} = M.$$

Then

$$Q^{*}\,\mu^{-1}\,Q = M\,Q = D,$$

a diagonal matrix, and the Q polynomials are summationwise orthogonal with respect to the given weighting matrix. The proof is that for $j > i$ the j^{th} mean difference of an i^{th} degree polynomial is zero.

The weighted least squares solution of $U \vec{A} = \vec{F}$ is the solution of

$$U^* \mu^{-1} U \vec{A} = U^* \mu^{-1} \vec{F}.$$

But

$$P_0 = b_{00}Q_0 \quad ,$$
$$P_1 = b_{01}Q_0 + b_{11}Q_1 \quad ,$$
$$- - - - - - - - - -$$
$$P_n = b_{0n}Q_0 + b_{1n}Q_1 + \dots b_{nn}Q_n \quad ,$$

or

$$U = Q\,B,$$

with B a nonsingular, triangular matrix. Therefore

$$U^* \mu^{-1} U \vec{A} = U^* \mu^{-1} \vec{F}$$

can be written

$$B^* Q^* \mu^{-1} U \vec{A} = B^* Q^* \mu^{-1} \vec{F}$$

or

$$B^* M U \vec{A} = B^* M \vec{F} \quad ,$$

and since B is nonsingular

$$M U \vec{A} = M \vec{F} \quad .$$

MEAN VALUES OF DIFFERENCES OVER A DISCRETE SET

The problem of finding the mean k^{th} difference from a set of functional values, defined over a discrete set, has been discussed in reference (2). A development, similar to that indicated in equation (4), is immediately seen by writing for r+1 equally spaced data, with k = 2 (see reference (2)), the weighted mean second difference equals

$$\overline{(\Delta^2 y)} = \frac{1}{\Sigma w}\Big[w_1(y_0 - 2y_1 + y_2) + w_2(y_1 - 2y_2 + y_3)$$
$$+ - - - - - - - - - - - - - - - - - - -$$
$$+ w_{r-1}(y_{r-2} - 2y_{r-1} + y_r)\Big] .$$

In matrix language this becomes

$$\overline{(\Delta^2 y)} = \frac{1}{\Sigma w} \begin{bmatrix} w_1 w_2 & \cdots & w_{r-1} \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -2 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_r \end{bmatrix} .$$

This expression for the mean second difference shows the w matrix multiplying a column matrix whose elements are second differences of y. To make the development parallel to that in equations (5) it is necessary to give the mean second difference in the form of a row matrix whose elements are second differences of a w set multiplying a column matrix whose elements are the y values. This is done by adding two zeros before and two zeros after the w's in the above matrix equation and adding rows above and below in the difference matrix to show differencing of the new w set.

$$\overline{(\Delta^2 y)} = \frac{1}{\Sigma w}\begin{bmatrix} 0 & 0 & w_1 & \ldots & w_{r-1} & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ \cdot \\ \cdot \\ y_r \end{bmatrix}.$$

With m's being the second differences of the new w set the equation now becomes

$$\overline{(\Delta^2 y)} = \frac{1}{\Sigma w}\begin{bmatrix} m_0 m_1 m_2 \cdots m_r \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_r \end{bmatrix} \tag{20}$$

Reference (3) makes the steps to equation (20) parallel to those of equation (4) by using summation by parts. For finding the mean k^{th} difference, the steps leading to equation (20) show that the m's are $(-1)^k$ times the k^{th} differences of the set

$$[000 \ldots 0 w_1 w_2 \ldots w_{r+1-k} 0 \ldots 00]$$

having k zeros at each end (reference (2)).

The Gram polynomials, which are the m's for equally spaced, equally weighted data, have been discussed in many sources. A partial tabulation may be found in reference (2) and in many other published works on least squares fitting. An interesting game, for any computer, is to see how far one can go in number of equally spaced points and degree of polynomial with the m's represented exactly by the number of digits used in single precision computation. (See Appendix A.)

When this limit is exceeded, the w's

$$cw_i = \binom{k-1+i}{k}\binom{r+1-i}{k} \tag{21}$$

can be cut in number of digits, and these approximate w's used to find the m's within the capacity of single precision arithmetic.

By considering the weighting function for getting the mean k^{th} derivative which leads to the development of the Jacobi polynomials, it is possible to construct a corresponding set of weights for finding the mean k^{th} difference. Take

$$cw = (1-x)^{\alpha+k}(1+x)^{\beta+k}$$

$$= (1-x)^{k(1+\frac{\alpha}{k})}(1+x)^{k(1+\frac{\beta}{k})}$$

$$= \left[(1-x)^k\right]^{(1+\frac{\alpha}{k})}\left[(1+x)^k\right]^{(1+\frac{\beta}{k})}, \alpha > -1, \beta > -1.$$

When this form is compared with the $\alpha = 0$, $\beta = 0$ case, which leads to the Legendre polynomials, to the weights

$$cw_i = \binom{r+1-i}{k}\binom{k-1+i}{k} ,$$

used for equally spaced, equally weighted, data in finding the mean

k^{th} difference corresponding to the k^{th} Gram polynomial, it is seen that the set of weights,

$$cw_i = \left[\binom{r+1-i}{k}\right]^{\left(1+\frac{\alpha}{k}\right)} \left[\binom{k-1+i}{k}\right]^{\left(1+\frac{\beta}{k}\right)} \quad ,k > 0 \qquad (22)$$

can be used for equally spaced discrete data in finding fits corresponding, roughly, to the least-squares fits made with Jacobi polynomials over a continuous set. It should be noted that for $k = 0$ the w's become indeterminate; for this case, rounded values of the elements of the diagonal matrix D, in equation (17), can be used.

In particular, corresponding to the Chebyshev case, if $\alpha = \beta = -1/2$,

$$cw_i = \left[\binom{r+1-i}{k}\right]^{\left(1-\frac{1}{2k}\right)} \left[\binom{k-1+i}{k}\right]^{\left(1-\frac{1}{2k}\right)}$$

$$= \left[\binom{k-1+i}{k}\binom{r+1-i}{k}\right]^{\left(\frac{2k-1}{2k}\right)} \quad , k > 0 \qquad (23)$$

These w's can easily be constructed from those tabulated in reference (2). With this in mind, equation (23) can be written

$$(cw_i)_T = \left[(w_i)_G\right]^{\left(\frac{2k-1}{2k}\right)} \qquad k > 0 \qquad (24)$$

The T subscript indicates fitting with Chebyshev type weights and the G subscript indicates fitting with Gram (or Legendre) type weights. The $(cw_i)_T$ values of equation (24) are, in general, irrational. Therefore, they must be rounded before k^{th} differences are taken to find the exact m's corresponding to the approximate w's. In general, if three significant digits are retained in the first (smallest) nonzero w, a satisfactory set of m's can be found.

The indeterminate form for equation (22) when k = 0 and $\alpha^2 + \beta^2 > 0$ can be avoided by writing, for the continuous case,

$$cw = (1-x)^{k-1} \quad (1-x)^{1+\alpha} \quad (1+x)^{k-1} \quad (1+x)^{1+\beta}, \quad k > 0,$$

$$cw = (1-x)^{\alpha} \ (1+x)^{\beta}, \qquad k = 0.$$

Equation (22) is then replaced by

$$cw_i = \binom{r+1-i}{k-1}\binom{r+2-k-i}{1}^{1+\alpha}\binom{k-1+i}{k-1}\binom{i}{1}^{1+\beta}, \text{for } k > 0,$$

and (22a)

$$cw_i = (r + 2 - k - i)^{\alpha} \quad (i)^{\beta}, \qquad \text{for } k = 0$$

The $\binom{r+2-k-i}{1}^{1+\alpha} \binom{i}{1}^{1+\beta}$, $(r + 2 - k - i)^{\alpha}$, $(i)^{\beta}$ are to be evaluated only when $r = 2 - k - i > 0$ and $i > 0$.

Note that for all cases corresponding to Jacobi type fits, $1 + \alpha > 0$ and $1 + \beta > 0$.

FITTING EQUALLY SPACED DISCRETE DATA WITH CERTAIN TYPE CONSTRAINTS

Reference (4) discusses power series approximations of functions having the d'Alembert characteristic. Special cases are:

$$f_1 = A_1x + A_3x^3 + A_5x^5 + \dots ,$$

$$f_4 = A_4x^4 + A_6x^6 + A_8x^8 + \dots ,$$

$$f_5 = A_5x^5 + A_7x^7 + A_9x^9 + \dots .$$

These are functions having a set of derivatives specified at a point. The m's, computed from a suitable set of w's, can be used for making polynomial approximations of functions of this type.

Suppose a transformation has been made in the independent variable so that u = 0 for the value of x for which certain derivatives are specified. The complete function can be approximated by

$$f(u) = A_0 + A_1u + A_2u^2 + \dots + A_nu^n . \qquad (25)$$

Suppose $f(0)$, $f'(0)$, $f'''(0)$, . . . $f^{(r)}(0)$ are specified. From

$$f(0) = A_0$$

$$f'(0) = A_1$$

$$\vdots$$

$$\frac{f^{(r)}(0)}{r!} = A_r$$

it is seen that the problem is that of fitting

$$g(u) = f(u) - A_0 - A_1 u - \ldots - A_r u^r$$

$$= A_2 u^2 + A_4 u^4 + \ldots A_n u^n \quad , \qquad (26)$$

and that the unspecified A's are to be chosen in a way consistent with the least-squares weighting.

Substituting in equation (26),

$$\begin{bmatrix} U_2 & U_4 & \cdots & U_n \end{bmatrix} \begin{bmatrix} A_2 \\ A_4 \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{bmatrix} = \vec{G}, \qquad (27)$$

$$U_2 = \begin{bmatrix} u_1^2 \\ u_2^2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} , \ldots . U_n = \begin{bmatrix} u_1^n \\ u_2^n \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

$$\vec{G} = \begin{bmatrix} g(u_1) \\ g(u_2) \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

If the matrix M is constructed by making the first row the m's for k = 2, the second row the m's for k = 4, ..., the last row the m's for k = n, premultiplying both sides of equation (27) by M will give the required number of equations for solving for A_2, A_4, ..., A_n, and the matrix of the coefficients of the A's will be a triangular, nonsingular, matrix. The method can be used when any of the A's of equation (1) are specified.

FITTING A FUNCTION OF TWO INDEPENDENT VARIABLES

Let it be supposed that values of the function are known for a rectangular grid, equally spaced in both independent variables. Make linear transformations in the independent variables so that the new independent variables become u and v.

Approximate f by

$$f = \begin{bmatrix} P_0(u) & P_1(u) & P_2(u) & \dots & P_\mu(u) \end{bmatrix} A \begin{bmatrix} P_0(v) \\ P_1(v) \\ P_2(v) \\ \cdot \\ \cdot \\ \cdot \\ P_\nu(v) \end{bmatrix} \qquad (28)$$

with

$$A = \begin{bmatrix} A_{00} & A_{01} & A_{02} & \cdots\cdots & A_{0\nu} \\ A_{10} & A_{11} & A_{12} & & A_{1\nu} \\ A_{20} & A_{21} & A_{22} & & A_{2\nu} \\ \cdot & \cdot & \cdot & \cdots\cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots\cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots\cdots & \cdot \\ A_{\mu 0} & A_{\mu 1} & A_{\mu 2} & & A_{\mu\nu} \end{bmatrix}$$

Construct

$$(MM')^{-1} M_u = (T^{-1} M)_u$$

for the number of u values in the grid and the degree in u desired.

as in equation (19). In the like manner construct

$$(\Omega V)^{-1} M_v = (T^{-1} M)_v$$

for the number of v values and degree in v desired. The A matrix, the coefficient matrix, is then

$$A = (T^{-1} M)_u \; F \; (T^{-1} M)_v^* \qquad (29)$$

the * indicating the transpose of the indicated matrix and

$$F = \begin{bmatrix} f(u_o, v_o) & f(u_o, v_1) & \cdots & f(u_o, v_s) \\ f(u_1, v_o) & f(u_1, v_1) & \cdots & f(u_1, vs) \\ \vdots & \vdots & \cdots & \vdots \\ f(u_r, v_o & f(u_r, v_1 & \cdots & f(u_r, v_s) \end{bmatrix}$$

It should be noted that the restriction that the rectangular grid be equally spaced can be removed if the unequally spaced grid in u and/or v is to be used frequently enough to justify the computation of the multipliers using divided differences. After computation of the multipliers equation (29) can be used. For unequal spacing and the resulting divided differences it may no longer be possible to go from the w's to the m's without round-off,

so that eight digit, ten digit, fifteen digit,... approximations will result with corresponding round-off errors in the A matrix.

END POINT SMOOTHING AND PREDICTION FROM EQUALLY SPACED DISCRETE DATA

The w function, $cw = x^k e^{-\alpha x}$, for finding the mean k^{th} derivative over the semi-infinite interval, $0 \leq x \leq \infty$, which leads to the k^{th} Laguerre polynomial, indicates a w form that can be used to solve the problem of end-point smoothing and prediction from equally spaced discrete data.

Suppose that the function, y, is known for u = 0, 1, 2, 3,... . Take

$$cw = \binom{k+u}{k} e^{-\alpha u} \quad .$$

Although the Laguerre polynomials are based on the semi-infinite interval, $0 \leq x < \infty$, and the above indicated u-set would go to infinity, in actual practice a finite number of points would be used.

For $k > 0$, the w's would be modified by replacing the last few w's with multiples of those used in Jacobi-type fittings. This "splicing" would be done so as to bring a gradual descent of the w's to zero and avoid unwanted magnitudes of the resulting m's.

A set of w's suitable for the solution of this problem can also be based on the continuous data w's for Jacobi polynomials. The set is formed by choosing $\beta \leq 0$ and $\alpha > 0$ in equation (22) for left-end smoothing; for right-end smoothing reverse the order of the nonzero w's.

One application of end-point smoothing is the problem of finding the time of motor separation from observed positions of a rocket-propelled missile. The difference between the right-hand smoothed estimate of $(\text{acceleration})_i$ and the left hand smoothed estimate of $(\text{acceleration})_{i+1}$ would be expected to have maximum magnitude when separation takes place between t_i and t_{i+1}.

Another application is that of aiming at a moving target. In this case the best possible estimate of present position vector and its derivatives is desired.

MID-POINT SMOOTHING FOR AN ODD NUMBER OF EQUALLY SPACED DATA POINTS

The w function, $cw = e^{-\alpha^2 x^2}$, for finding the mean k^{th} derivative over the infinite interval $-\alpha < x < \infty$ can be used for this problem. For discrete data the first estimate of the w's would be generated from $cw = e^{-\alpha^2 u^2}$ with $u = -m, \ldots -3, -2, -1, 0, 1, 2, 3, \ldots, m$ and final w's can be obtained by replacing at

either end with w's from a Jacobi-type fit. The need for this modification is not so severe as in the case of end-point smoothing.

This problem can also be solved by using equation (22) and setting $\alpha = \beta > 0$.

REFERENCES

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(3) F. B. Hildebrand, Introduction to Numerical Analysis, McGraw-Hill Book Company, Inc., 1956.

(4) Dirk Brouwer and Gerald M. Clemence, Methods of Celestial Mechanics, Academic Press, 1961.

APPENDIX A

COMPUTATION OF w AND m VALUES BY HIGH SPEED COMPUTER

For computational purposes, every number whose factorial must be evaluated is represented in its prime factor form. This is necessary in order to conveniently represent the w's in the equation,

$$c(w_k)_i = \frac{(k - 1 + i)!}{k!\,(i - 1)!} \cdot \frac{(n - i)!}{k!\,(n - i - k)!} \qquad (1)$$

where

n = number of data points

and

k = the order of difference.

If the computations were performed in decimal form, the resulting value of $(w_k)_i$ might well be expected to exceed the IBM 7030's maximum word length of 14 digits. The product of the numerators of equation (1) may be formed by adding the exponents of the corresponding prime factors of each. Likewise, the product of the denominators may be formed in the same way. Division may be accomplished by subtraction of the exponents of the corresponding prime factors of each, resulting in a value for $c(w_k)_i$ which is the product of 15* prime factors raised to various powers.

---

* The number 15 was chosen because in the computer program 15 prime factors are needed to represent the factorials of all the numbers from 1 to 50. They are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, and 47.

After the (n - k) values of $c(w_k)_i$ are computed, the $(w_k)_i$'s are reduced to their lowest terms. Then, evaluation of the decimal equivalent of each prime factor $(w_k)_i$ can be made. This is accomplished by first obtaining 5 decimal numbers, each number computed by forming the product of 3 of the 15 quantities in the prime factor representation of $(w_k)_i$. The following illustration may clarify the procedure:

$$(w_k)_i = 2^{Q_1} \cdot 3^{Q_2} \cdot 5^{Q_3} \cdot 7^{Q_4} \cdot 11^{Q_5} \cdot 13^{Q_6} \cdot 17^{Q_7} \cdot 19^{Q_8} \cdot 23^{Q_9}$$

$$\cdot\ 29^{Q_{10}} \cdot 31^{Q_{11}} \cdot 37^{Q_{12}} \cdot 41^{Q_{13}} \cdot 43^{Q_{14}} \cdot 47^{Q_{15}} \quad ,$$

where Q_m is an integer or zero (m = 1, 2 . . . 15):

$$(w_k)_i = (2^{Q_1} \cdot 13^{Q_6} \cdot 31^{Q_{11}}) \cdot (3^{Q_2} \cdot 17^{Q_7} \cdot 37^{Q_{12}}) \cdot \cdot \cdot \cdot$$

$$(11^{Q_5} \cdot 29^{Q_{10}} \cdot 47^{Q_{15}})$$

$$= R_1 \cdot R_2 \cdot \cdot \cdot \cdot R_M \ , \ (M = 1, 2, \ldots 5),$$

where each expression in parentheses (and therefore $R_1 \cdot R_2 \cdot \cdot \cdot R_5$) is a decimal number.

At this point, each R_M is examined and in every case the number of places to the left of the decimal of each R_M is stored. These "magnitude tags" are then summed and the result is an indication of the magnitude of each $(w_k)_i$. If the sum of the "magnitude tags" is less than 15, multiplication of the R_M's is performed and a $(w_k)_i$ is obtained. If the sum of the "magnitude tags" is 15 or more, R_1 is repeatedly divided

by 10 until the sum of the "magnitude tags" indicates that the new (reduced) $(w_k)_i$ can be represented in less than 15 digits. The R_M's are multiplied and each $(w_k)_i$ value obtained is truncated to an integer.

Thus, having computed the $(w_k)_i$'s, the next step is to obtain the m's. This is done by evaluating the matrix equation, $\vec{m} = [B][\vec{w}]$, where

$$\vec{m} = \begin{bmatrix} (m_k)_1 \\ (m_k)_2 \\ \cdot \\ \cdot \\ \cdot \\ (m_k)_n \end{bmatrix},$$

$$B = \begin{bmatrix} x_1 & x_2 & x_3 & \cdot & \cdot & \cdot & x_{k+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & x_2 & \cdot & \cdot & \cdot & \cdot & x_{k+1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & \cdot & \cdot & \cdot & \cdot & \cdot & x_{k+1} & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & \cdot & \cdot & \cdot & x_{k+1} \end{bmatrix},$$

and

$$\vec{w} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ (w_k)_1 \\ (w_k)_2 \\ \cdot \\ \cdot \\ \cdot \\ (w_k)_n \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{bmatrix}$$

with k zeros at each end.

That each $(m_k)_i$ may be contained in less than 15 digits is assured by employing another "magnitude tag" before each multiplication.

APPENDIX B

TABLES OF TRANSPOSE OF M AND INVERSE OF MU

These tables are designed for finding the weighted least squares fits in the form

$$y = A_0 + A_1 u + A_2 u^2 + A_3 u^3 + A_4 u^4 + A_5 u^5$$

to 2s + 1 data points, u taking the values

$$-s, (-s + 1), \ldots -2, -1, 0, 1, 2, \ldots s - 1, s.$$

The w's used in constructing the M matrices were constructed by ordering the distinct non-zero computed w's so that

$$w_1 \leq w_2 \leq w_3 \leq \cdots w_m$$

and choosing the q from the integers 1, 2, 3, ... N, N $\leq$ 25, which makes

$$\sum_{j=2}^{m} \left(\frac{1}{w_1} - \frac{p_j}{q w_j} \right)^2$$

a minimum, p_j being the integer nearest $\left(\frac{q}{w_1} \right) w_j$.

The computed $w_1, w_2, \ldots w_m$ values were then replaced with

$$p_1 = q, p_2, p_3, \ldots p_m.$$

Weights were chosen so as to make the resulting fits correspond, [illegible], to least squares fits using Jacobi polynomials with weighting indicated by the given (α , β) values. Thus (0,0)5 indicates that a 5th degree fit is being made with the "Jacobi" (α . β) equal to

(0,0). This is the least squares fit with uniform, or equal, weighting. (-1/2, -1/2)5 indicates that a fifth degree fit is being made with the "Jacobi" (α, β) equal to (-1/2, -1/2). This corresponds to the Chebyshev, or minimax, type fit, and is a least squares fit with heavy weighting at the ends and smaller weighting at the center of the interval. (1/2, 1/2)5 indicates that a fifth degree fit is being made with the "Jacobi" (α, β) equal to (1/2, 1/2). This is a least squares fit with heavy weighting at the center and smaller weighting at the ends of the interval.

Since the u values and the weighting chosen have symmetry about the mid point, or center, each $(MU)^{-1}$ takes the form.

$$(MU)^{-1} = \begin{bmatrix} c_{00} & 0 & c_{02} & 0 & c_{04} & 0 \\ 0 & c_{11} & 0 & c_{13} & 0 & c_{15} \\ 0 & 0 & c_{22} & 0 & c_{24} & 0 \\ 0 & 0 & 0 & c_{33} & 0 & c_{35} \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{55} \end{bmatrix}$$

Let $\vec{F}$ be the vector defined for equation (17) with $u_0 = -s$, $u_1 = -s + 1$, ... and $u_m = s$, and

$\vec{m}_0$ be the first column of M^*,

$\vec{m}_1$ be the second column of M*,

- - - - - - - - - - - - - - - -

$\vec{m}_5$ be the sixth column of M* .

The A's in the least squares polynomial approximation of the function are then given by

$$A_0 = C_{00}\,\vec{m}_0 \cdot \vec{F} + C_{02}\,\vec{m}_2 \cdot \vec{F} + C_{04}\,\vec{m}_4 \cdot \vec{F}$$
$$A_1 = C_{11}\,\vec{m}_1 \cdot \vec{F} + C_{13}\,\vec{m}_3 \cdot \vec{F} + C_{15}\,\vec{m}_5 \cdot \vec{F}$$
$$A_2 = C_{22}\,\vec{m}_2 \cdot \vec{F} + C_{24}\,\vec{m}_4 \cdot \vec{F}$$
$$A_3 = C_{33}\,\vec{m}_3 \cdot \vec{F} + C_{35}\,\vec{m}_5 \cdot \vec{F}$$
$$A_4 = C_{44}\,\vec{m}_4 \cdot \vec{F}$$
$$A_5 = C_{55}\,\vec{m}_5 \cdot \vec{F}$$

Lower degree fits can be made by deleting corresponding columns, from the right, in the matrices M* and $(MU)^{-1}$.

(0,0)5 11 Points

$$(MU)^{-1} = \begin{bmatrix} \frac{1}{11} & 0 & \frac{-5}{286} & 0 & \frac{853}{40898} & 0 \\ 0 & \frac{1}{110} & 0 & \frac{-89}{8580} & 0 & \frac{1141}{37180} \\ 0 & 0 & \frac{1}{572} & 0 & \frac{-3569}{490776} & 0 \\ 0 & 0 & 0 & \frac{1}{1716} & 0 & \frac{-907}{178464} \\ 0 & 0 & 0 & 0 & \frac{1}{3432} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6240} \end{bmatrix}$$

(0,0)5 11 Points

$$M^* = \begin{bmatrix} 1 & -5 & 10 & -10 & 6 & -3 \\ 1 & -4 & 4 & 2 & -6 & 6 \\ 1 & -3 & -1 & 7 & -6 & 1 \\ 1 & -2 & -3 & 9 & -1 & -4 \\ 1 & -1 & -7 & 3 & 4 & -4 \\ 1 & 0 & -6 & 0 & 6 & 0 \\ 1 & 1 & -7 & -3 & 4 & 4 \\ 1 & 2 & -3 & -9 & -1 & 4 \\ 1 & 3 & -1 & -7 & -6 & -1 \\ 1 & 4 & 4 & -2 & -6 & -6 \\ 1 & 5 & 10 & 10 & 6 & 3 \end{bmatrix}$$

(-1/2, -1/2) 5 11 Points

$$(MU)^{-1} = \begin{bmatrix} \frac{1}{166} & 0 & \frac{-1}{39} & 0 & \frac{27323}{1437228} & 0 \\ 0 & \frac{1}{162} & 0 & \frac{-7}{500} & 0 & \frac{40753}{931500} \\ 0 & 0 & \frac{1}{468} & 0 & \frac{-1081}{207792} & 0 \\ 0 & 0 & 0 & \frac{1}{1500} & 0 & \frac{-287}{46000} \\ 0 & 0 & 0 & 0 & \frac{1}{5328} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5520} \end{bmatrix}$$

(-1/2, -1/2) 5 11 Points

$$M^* = \begin{bmatrix} 22 & -11 & 11 & -11 & 11 & -3 \\ 16 & -4 & -1 & 7 & -15 & 7 \\ 14 & -2 & -1 & 5 & -5 & -2 \\ 13 & -2 & -3 & 5 & 2 & -2 \\ 12 & 0 & -4 & 2 & 2 & -3 \\ 12 & 0 & -4 & 0 & 10 & 0 \\ 12 & 0 & -4 & -2 & 2 & 3 \\ 13 & 2 & -3 & -5 & 2 & 2 \\ 14 & 2 & -1 & -5 & -5 & 2 \\ 16 & 4 & -1 & -7 & -15 & -7 \\ 22 & 11 & 11 & 11 & 11 & 3 \end{bmatrix}$$

(1/2, 1/2)5 11 Points

$$
(MU)^{-1} = \begin{bmatrix}
\frac{1}{148} & 0 & \frac{-619}{58756} & 0 & \frac{7775}{940096} & 0 \\
0 & \frac{1}{312} & 0 & \frac{-131}{11336} & 0 & \frac{110159}{14963520} \\
0 & 0 & \frac{1}{794} & 0 & \frac{-8881}{2439168} & 0 \\
0 & 0 & 0 & \frac{1}{1308} & 0 & \frac{-637}{460416} \\
0 & 0 & 0 & 0 & \frac{1}{6144} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{21120}
\end{bmatrix}
$$

(1/2, 1/2)5 11 Points

$$
M^{*} = \begin{bmatrix}
9 & -9 & 10 & -6 & 9 & -9 \\
12 & -13 & 10 & -2 & -5 & 15 \\
14 & -12 & 2 & 6 & -15 & 11 \\
15 & -9 & -5 & 7 & -4 & -14 \\
16 & -5 & -11 & 6 & 6 & -20 \\
16 & 0 & -12 & 0 & 18 & 0 \\
16 & 5 & -11 & -6 & 6 & 20 \\
15 & 9 & -5 & -7 & -4 & 14 \\
14 & 12 & 2 & -6 & -15 & -11 \\
12 & 13 & 10 & 2 & -5 & -15 \\
9 & 9 & 10 & 6 & 9 & 9
\end{bmatrix}
$$

(0,0)5 15 Points

$$(MU)^{-1} = \begin{bmatrix} \frac{1}{15} & 0 & \frac{-28}{2859} & 0 & \frac{154546}{9248865} & 0 \\ 0 & \frac{1}{280} & 0 & \frac{-167}{39360} & 0 & \frac{80611}{16173024} \\ 0 & 0 & \frac{1}{1906} & 0 & \frac{-45077}{14798184} & 0 \\ 0 & 0 & 0 & \frac{1}{7872} & 0 & \frac{-331}{770144} \\ 0 & 0 & 0 & 0 & \frac{1}{15528} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{140880} \end{bmatrix}$$

(0,0)5 15 Points

$$M^* = \begin{bmatrix} 1 & -7 & 14 & -15 & 7 & -14 \\ 1 & -6 & 8 & -2 & -3 & 16 \\ 1 & -5 & 3 & 5 & -6 & 14 \\ 1 & -4 & -1 & 11 & -5 & -1 \\ 1 & -3 & -5 & 9 & -2 & -7 \\ 1 & -2 & -7 & 8 & 2 & -18 \\ 1 & -1 & -7 & 5 & 5 & -7 \\ 1 & 0 & -10 & 0 & 4 & 0 \\ 1 & 1 & -7 & -5 & 5 & 7 \\ 1 & 2 & -7 & -8 & 2 & 18 \\ 1 & 3 & -5 & -9 & -2 & 7 \\ 1 & 4 & -1 & -11 & -5 & 1 \\ 1 & 5 & 3 & -5 & -6 & -14 \\ 1 & 6 & 8 & 2 & -3 & -16 \\ 1 & 7 & 14 & 15 & 7 & 14 \end{bmatrix}$$

(-1/2, -1/2)5 15 Points

$$(MU)^{-1} = \begin{bmatrix} \frac{1}{268} & 0 & \frac{-193}{6633} & 0 & \frac{423473}{25603380} & 0 \\ 0 & \frac{1}{304} & 0 & \frac{-763}{104196} & 0 & \frac{2645549}{252848960} \\ 0 & 0 & \frac{1}{792} & 0 & \frac{-1763}{764280} & 0 \\ 0 & 0 & 0 & \frac{1}{5484} & 0 & \frac{-6113}{7984704} \\ 0 & 0 & 0 & 0 & \frac{1}{23160} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{87360} \end{bmatrix}$$

(-1/2, -1/2)5 15 Points

$$M^* = \begin{bmatrix} 29 & -13 & 9 & -15 & 14 & -11 \\ 21 & -5 & 0 & 6 & -13 & 18 \\ 18 & -3 & 0 & 5 & -7 & 1 \\ 16 & -2 & -1 & 5 & -3 & 0 \\ 15 & -2 & -2 & 5 & -1 & -7 \\ 14 & -1 & -2 & 3 & 2 & -6 \\ 14 & 0 & -3 & 3 & 6 & -3 \\ 14 & 0 & -2 & 0 & 4 & 0 \\ 14 & 0 & -3 & -3 & 6 & 3 \\ 14 & 1 & -2 & -3 & 2 & 6 \\ 15 & 2 & -2 & -5 & -1 & 7 \\ 16 & 2 & -1 & -5 & -3 & 0 \\ 18 & 3 & 0 & -5 & -7 & -1 \\ 21 & 5 & 0 & -6 & -13 & -18 \\ 29 & 13 & 9 & 15 & 14 & 11 \end{bmatrix}$$

(1/2, 1/2) 5 15 Points

$$(MU)^{-1} = \begin{bmatrix} \frac{1}{411} & 0 & \frac{-3130}{662943} & 0 & \frac{2515439}{617862876} & 0 \\ 0 & \frac{1}{930} & 0 & \frac{-4373}{1294560} & 0 & \frac{758251}{206805960} \\ 0 & 0 & \frac{1}{3226} & 0 & \frac{-67553}{72159168} & 0 \\ 0 & 0 & 0 & \frac{1}{8352} & 0 & \frac{-961}{2668464} \\ 0 & 0 & 0 & 0 & \frac{1}{44736} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{153360} \end{bmatrix}$$

(1/2, 1/2) 5 15 Points

$$M^* = \begin{bmatrix} 16 & -13 & 15 & -11 & 15 & -12 \\ 22 & -20 & 19 & -9 & 3 & 7 \\ 26 & -21 & 11 & 3 & -19 & 22 \\ 29 & -19 & 3 & 11 & -19 & 5 \\ 31 & -15 & -6 & 13 & -11 & -11 \\ 32 & -11 & -14 & 13 & 5 & -19 \\ 33 & -6 & -19 & 7 & 15 & -17 \\ 33 & 0 & -18 & 0 & 22 & 0 \\ 33 & 6 & -19 & -7 & 15 & 17 \\ 32 & 11 & -14 & -13 & 5 & 19 \\ 31 & 15 & -6 & -13 & -11 & 11 \\ 29 & 19 & 3 & -11 & -19 & -5 \\ 26 & 21 & 11 & -3 & -19 & -22 \\ 22 & 20 & 19 & 9 & 3 & -7 \\ 16 & 13 & 15 & 11 & 15 & 12 \end{bmatrix}$$

(0,0)5 19 Points

$$(MU)^{-1} = \begin{bmatrix} \frac{1}{19} & 0 & \frac{-15}{2396} & 0 & \frac{1479}{250382} & 0 \\ 0 & \frac{1}{570} & 0 & \frac{-269}{94020} & 0 & \frac{363581}{123260220} \\ 0 & 0 & \frac{1}{4792} & 0 & \frac{-45833}{69105432} & 0 \\ 0 & 0 & 0 & \frac{1}{18804} & 0 & \frac{-30811}{197216352} \\ 0 & 0 & 0 & 0 & \frac{1}{115368} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{629280} \end{bmatrix}$$

(0,0)5 19 Points

$$M^* = \begin{bmatrix} 1 & -9 & 18 & -15 & 18 & -18 \\ 1 & -8 & 12 & -5 & -2 & 12 \\ 1 & -7 & 7 & 2 & -11 & 17 \\ 1 & -6 & 2 & 7 & -15 & 12 \\ 1 & -5 & -2 & 8 & -8 & -5 \\ 1 & -4 & -5 & 10 & -6 & -3 \\ 1 & -3 & -8 & 7 & 0 & -21 \\ 1 & -2 & -8 & 8 & 10 & -6 \\ 1 & -1 & -11 & 2 & 5 & -13 \\ 1 & 0 & -10 & 0 & 18 & 0 \\ 1 & 1 & -11 & -2 & 5 & 13 \\ 1 & 2 & -8 & -8 & 10 & 6 \\ 1 & 3 & -8 & -7 & 0 & 21 \\ 1 & 4 & -5 & -10 & -6 & 3 \\ 1 & 5 & -2 & -8 & -8 & 5 \\ 1 & 6 & 2 & -7 & -15 & -12 \\ 1 & 7 & 7 & -2 & -11 & -17 \\ 1 & 8 & 12 & 5 & -2 & -12 \\ 1 & 9 & 18 & 15 & 18 & 18 \end{bmatrix}$$

(-1/2, -1/2)5 19 Points

$$
(MU)^{-1} = \begin{bmatrix}
\frac{1}{496} & 0 & \frac{-4663}{346808} & 0 & \frac{31834717}{1842280400} & 0 \\
0 & \frac{1}{630} & 0 & \frac{-403}{73080} & 0 & \frac{29885}{4881744} \\
0 & 0 & \frac{1}{2942} & 0 & \frac{-129079}{89142600} & 0 \\
0 & 0 & 0 & \frac{1}{11832} & 0 & \frac{-10799}{39518880} \\
0 & 0 & 0 & 0 & \frac{1}{60600} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{400800}
\end{bmatrix}
$$

(-1/2, -1/2)5 19 Points

$$
M^* = \begin{bmatrix}
46 & -19 & 19 & -15 & 14 & -16 \\
33 & -7 & 2 & 4 & -10 & 20 \\
28 & -5 & 0 & 3 & -5 & 5 \\
25 & -4 & -1 & 5 & -5 & 2 \\
23 & -3 & -2 & 4 & -3 & -1 \\
22 & -2 & -3 & 4 & -2 & -5 \\
21 & -1 & -5 & 3 & 3 & -9 \\
20 & -1 & -3 & 3 & 2 & -4 \\
20 & -1 & -5 & 1 & 3 & -3 \\
20 & 0 & -4 & 0 & 6 & 0 \\
20 & 1 & -5 & -1 & 3 & 3 \\
20 & 1 & -3 & -3 & 2 & 4 \\
21 & 1 & -5 & -3 & 3 & 9 \\
22 & 2 & -3 & -4 & -2 & 5 \\
23 & 3 & -2 & -4 & -3 & 1 \\
25 & 4 & -1 & -5 & -5 & -2 \\
28 & 5 & 0 & -3 & -5 & -5 \\
33 & 7 & 2 & -4 & -10 & -20 \\
46 & 19 & 19 & 15 & 14 & 16
\end{bmatrix}
$$

(1/2, 1/2)5 19 Points

$$
(MU)^{-1} = \begin{bmatrix}
\frac{1}{678} & 0 & \frac{-1633}{539010} & 0 & \frac{142657}{51421554} & 0 \\
0 & \frac{1}{2270} & 0 & \frac{-51427}{29664360} & 0 & \frac{45610211}{10582760430} \\
0 & 0 & \frac{1}{7950} & 0 & \frac{-17869}{45505800} & 0 \\
0 & 0 & 0 & \frac{1}{26136} & 0 & \frac{-19429}{74592144} \\
0 & 0 & 0 & 0 & \frac{1}{171720} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{342480}
\end{bmatrix}
$$

(1/2, 1/2)5 19 Points

$$
M^{*} = \begin{bmatrix}
19 & -18 & 17 & -13 & 1\text{[illegible]} & -7 \\
26 & -29 & 24 & -14 & 11 & 0 \\
31 & -32 & 19 & -3 & -13 & 10 \\
35 & -31 & 11 & 6 & -22 & 10 \\
38 & -29 & 2 & 14 & -22 & 1 \\
40 & -24 & -6 & 16 & -16 & -4 \\
42 & -19 & -13 & 16 & 1 & -10 \\
43 & -13 & -20 & 12 & 11 & -10 \\
43 & -7 & -22 & 8 & 20 & -6 \\
44 & 0 & -24 & 0 & 24 & 0 \\
43 & 7 & -22 & -8 & 20 & 6 \\
43 & 13 & -20 & -12 & 11 & 10 \\
42 & 19 & -13 & -16 & 1 & 10 \\
40 & 24 & -6 & -16 & -16 & 4 \\
38 & 29 & 2 & -14 & -22 & -1 \\
35 & 31 & 11 & -6 & -22 & -10 \\
31 & 32 & 19 & 3 & -13 & -10 \\
26 & 29 & 24 & 14 & 11 & 0 \\
19 & 18 & 17 & 13 & 18 & 7
\end{bmatrix}
$$

(0,0)5 23 Points

$$
(MU)^{-1} = \begin{bmatrix}
\frac{1}{23} & 0 & \frac{-22}{3449} & 0 & \frac{211255}{48617104} & 0 \\
0 & \frac{1}{1012} & 0 & \frac{-79}{56388} & 0 & \frac{4752269}{2529283740} \\
0 & 0 & \frac{1}{6898} & 0 & \frac{-387701}{1166810496} & 0 \\
0 & 0 & 0 & \frac{1}{56388} & 0 & \frac{-45429}{674475664} \\
0 & 0 & 0 & 0 & \frac{1}{338304} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2153040}
\end{bmatrix}
$$

(0,0)5 23 Points

$$
M^{*} = \begin{bmatrix}
1 & -11 & 15 & -22 & 22 & -22 \\
1 & -10 & 11 & -10 & 2 & 8 \\
1 & -9 & 7 & -1 & -9 & 18 \\
1 & -8 & 4 & 6 & -16 & 16 \\
1 & -7 & 1 & 10 & -12 & 8 \\
1 & -6 & -2 & 12 & -12 & -1 \\
1 & -5 & -3 & 13 & -8 & -9 \\
1 & -4 & -6 & 12 & 3 & -15 \\
1 & -3 & -6 & 10 & 0 & -14 \\
1 & -2 & -8 & 7 & 14 & -11 \\
1 & -1 & -9 & 4 & 7 & -9 \\
1 & 0 & -8 & 0 & 18 & 0 \\
1 & 1 & -9 & -4 & 7 & 9 \\
1 & 2 & -8 & -7 & 14 & 11 \\
1 & 3 & -6 & -10 & 0 & 14 \\
1 & 4 & -6 & -12 & 3 & 15 \\
1 & 5 & -3 & -13 & -8 & 9 \\
1 & 6 & -2 & -12 & -12 & 1 \\
1 & 7 & 1 & -10 & -12 & -8 \\
1 & 8 & 4 & -6 & -16 & -16 \\
1 & 9 & 7 & 1 & -9 & -18 \\
1 & 10 & 11 & 10 & 2 & -8 \\
1 & 11 & 15 & 22 & 22 & 22
\end{bmatrix}
$$

(-1/2, -1/2)5 23 Points

$$(MU)^{-1} = \begin{bmatrix} \frac{1}{641} & 0 & \frac{-17965}{1408918} & 0 & \frac{44761603}{3923836630} & 0 \\ 0 & \frac{1}{826} & 0 & \frac{-39947}{9778188} & 0 & \frac{362289023}{100927197140} \\ 0 & 0 & \frac{1}{4396} & 0 & \frac{-142501}{220371480} & 0 \\ 0 & 0 & 0 & \frac{1}{23676} & 0 & \frac{-63901}{586501872} \\ 0 & 0 & 0 & 0 & \frac{1}{200520} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{1486320} \end{bmatrix}$$

(-1/2, -1/2)5 23 Points

$$M^* = \begin{bmatrix} 53 & -19 & 18 & -16 & 21 & -23 \\ 38 & -7 & 2 & 3 & -12 & 24 \\ 32 & -5 & 2 & 2 & -5 & 4 \\ 28 & -4 & -1 & 4 & -10 & 12 \\ 26 & -3 & -1 & 4 & -2 & 0 \\ 24 & -3 & -2 & 3 & -7 & -8 \\ 23 & -1 & -2 & 5 & 0 & 4 \\ 22 & -2 & -4 & 4 & 1 & -18 \\ 22 & -1 & -3 & 1 & 1 & 0 \\ 21 & -1 & -3 & 3 & 6 & -8 \\ 21 & 0 & -4 & 0 & 5 & -3 \\ 21 & 0 & -4 & 0 & 4 & 0 \\ 21 & 0 & -4 & 0 & 5 & 3 \\ 21 & 1 & -3 & -3 & 6 & 8 \\ 22 & 1 & -3 & -1 & 1 & 0 \\ 22 & 2 & -4 & -4 & 1 & 18 \\ 23 & 1 & -2 & -5 & 0 & -4 \\ 24 & 3 & -2 & -3 & -7 & 8 \\ 26 & 3 & -1 & -4 & -2 & 0 \\ 28 & 4 & -1 & -4 & -10 & -12 \\ 32 & 5 & 2 & -2 & -5 & -4 \\ 38 & 7 & 2 & -3 & -12 & -24 \\ 53 & 19 & 18 & 16 & 21 & 23 \end{bmatrix}$$

(1/2, 1/2)5 23 Points

$$(MU)^{-1} = \begin{bmatrix} \frac{1}{1076} & 0 & \frac{-18869}{6656136} & 0 & \frac{23798349}{9778973140} & 0 \\ 0 & \frac{1}{2798} & 0 & \frac{-92719}{75143088} & 0 & \frac{5006059409}{5121001447200} \\ 0 & 0 & \frac{1}{12372} & 0 & \frac{-101939}{436236720} & 0 \\ 0 & 0 & 0 & \frac{1}{53712} & 0 & \frac{-2935}{73209456} \\ 0 & 0 & 0 & 0 & \frac{1}{423120} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3271200} \end{bmatrix}$$

(1/2, 1/2)5 23 Points

$$M^* = \begin{bmatrix} 23 & -14 & 14 & -12 & 17 & -22 \\ 32 & -23 & 21 & -15 & 15 & -9 \\ 38 & -26 & 19 & -7 & -5 & 25 \\ 43 & -27 & 14 & 1 & -16 & 30 \\ 47 & -26 & 8 & 7 & -22 & 23 \\ 50 & -24 & 2 & 14 & -20 & 3 \\ 52 & -21 & -5 & 14 & -17 & -8 \\ 54 & -18 & -10 & 17 & 0 & -27 \\ 56 & -13 & -14 & 13 & 1 & -34 \\ 57 & -9 & -19 & 11 & 17 & -22 \\ 57 & -5 & -20 & 5 & 18 & -18 \\ 58 & 0 & -20 & 0 & 24 & 0 \\ 57 & 5 & -20 & -5 & 18 & 18 \\ 57 & 9 & -19 & -11 & 17 & 22 \\ 56 & 13 & -14 & -13 & 1 & 34 \\ 54 & 18 & -10 & -17 & 0 & 27 \\ 52 & 21 & -5 & -14 & -17 & 8 \\ 50 & 24 & 2 & -14 & -20 & -3 \\ 47 & 26 & 8 & -7 & -22 & -23 \\ 43 & 27 & 14 & -1 & -16 & -30 \\ 38 & 26 & 19 & 7 & -5 & -25 \\ 32 & 23 & 21 & 15 & 15 & 9 \\ 23 & 14 & 14 & 12 & 17 & 22 \end{bmatrix}$$

(0,0)5 27 Points

$$
(MU)^{-1} = \begin{bmatrix}
\frac{1}{27} & 0 & \frac{-91}{24117} & 0 & \frac{2824003}{682921089} & 0 \\
0 & \frac{1}{1638} & 0 & \frac{-109}{112764} & 0 & \frac{31854566}{18384901605} \\
0 & 0 & \frac{1}{16078} & 0 & \frac{-1248647}{5463368712} & 0 \\
0 & 0 & 0 & \frac{1}{112764} & 0 & \frac{-125675}{2801508816} \\
0 & 0 & 0 & 0 & \frac{1}{679608} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4471920}
\end{bmatrix}
$$

(0,0)5 27 Points

$$
M^* = \begin{bmatrix}
1 & -13 & 22 & -24 & 21 & -19 \\
1 & -12 & 17 & -13 & 5 & 3 \\
1 & -11 & 12 & -4 & -6 & 12 \\
1 & -10 & 8 & 3 & -11 & 16 \\
1 & -9 & 5 & 7 & -12 & 7 \\
1 & -8 & 0 & 12 & -14 & 8 \\
1 & -7 & -3 & 13 & -9 & -2 \\
1 & -6 & -4 & 13 & -6 & -7 \\
1 & -5 & -8 & 12 & -3 & -11 \\
1 & -4 & -9 & 13 & 3 & -10 \\
1 & -3 & -10 & 9 & 8 & -13 \\
1 & -2 & -12 & 5 & 6 & -5 \\
1 & -1 & -11 & 5 & 13 & -8 \\
1 & 0 & -14 & 0 & 10 & 0 \\
1 & 1 & -11 & -5 & 13 & 8 \\
1 & 2 & -12 & -5 & 6 & 5 \\
1 & 3 & -10 & -9 & 8 & 13 \\
1 & 4 & -9 & -13 & 3 & 10 \\
1 & 5 & -8 & -12 & -3 & 11 \\
1 & 6 & -4 & -13 & -6 & 7 \\
1 & 7 & -3 & -13 & -9 & 2 \\
1 & 8 & 0 & -12 & -14 & -8 \\
1 & 9 & 5 & -7 & -12 & -7 \\
1 & 10 & 8 & -3 & -11 & -16 \\
1 & 11 & 12 & 4 & -6 & -12 \\
1 & 12 & 17 & 13 & 5 & -3 \\
1 & 13 & 22 & 24 & 21 & 19
\end{bmatrix}
$$

(-1/2, -1/2)5 27 Points

$$
(MU)^{-1} = \begin{bmatrix}
\frac{1}{907} & 0 & \frac{-35291}{4090570} & 0 & \frac{749115721}{47814672730} & 0 \\
0 & \frac{1}{1388} & 0 & \frac{-11579}{5029071} & 0 & \frac{2143936001}{667760047380} \\
0 & 0 & \frac{1}{9020} & 0 & \frac{-405101}{632608680} & 0 \\
0 & 0 & 0 & \frac{1}{57972} & 0 & \frac{-72289}{1026336288} \\
0 & 0 & 0 & 0 & \frac{1}{280536} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3186720}
\end{bmatrix}
$$

(-1/2, -1/2)5 27 Points

$$
M^{*} = \begin{bmatrix}
67 & -25 & 25 & -23 & 15 & -22 \\
49 & -9 & 4 & 3 & -7 & 19 \\
40 & -7 & 2 & 2 & -3 & 5 \\
36 & -6 & 1 & 3 & -5 & 9 \\
33 & -4 & -1 & 6 & -5 & 5 \\
30 & -4 & -1 & 5 & -1 & -2 \\
29 & -3 & -3 & 6 & -7 & -3 \\
28 & -2 & -3 & 3 & 5 & 1 \\
27 & -2 & -5 & 6 & -5 & -14 \\
26 & -2 & -3 & 3 & 5 & 1 \\
26 & -1 & -4 & 4 & -1 & -9 \\
25 & -1 & -5 & 1 & 5 & -4 \\
25 & 0 & -4 & 1 & 2 & 0 \\
25 & 0 & -6 & 0 & 4 & 0 \\
25 & 0 & -4 & -1 & 2 & 0 \\
25 & 1 & -5 & -1 & 5 & 4 \\
26 & 1 & -4 & -4 & -1 & 9 \\
26 & 2 & -3 & -3 & 5 & -1 \\
27 & 2 & -5 & -6 & -5 & 14 \\
28 & 2 & -3 & -3 & 5 & -1 \\
29 & 3 & -3 & -6 & -7 & 3 \\
30 & 4 & -1 & -5 & -1 & 2 \\
33 & 4 & -1 & -6 & -5 & -5 \\
36 & 6 & 1 & -3 & -5 & -9 \\
40 & 7 & 2 & -2 & -3 & -5 \\
49 & 9 & 4 & -3 & -7 & -19 \\
67 & 25 & 25 & 23 & 15 & 22
\end{bmatrix}
$$

$(1/2,\ 1/2)5$ 27 Points

$$
(MU)^{-1} = \begin{bmatrix}
\frac{1}{1473} & 0 & \frac{-11787}{8537999} & 0 & \frac{2101491004}{901382168427} & 0 \\
0 & \frac{1}{6518} & 0 & \frac{-297583}{546925380} & 0 & \frac{6966467089}{10934679122340} \\
0 & 0 & \frac{1}{34778} & 0 & \frac{-2371345}{14686471176} & 0 \\
0 & 0 & 0 & \frac{1}{167820} & 0 & \frac{-33779}{1789453472} \\
0 & 0 & 0 & 0 & \frac{1}{844584} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{9596640}
\end{bmatrix}
$$

$(1/2,\ 1/2)5$ 27 Points

$$
M^{*} = \begin{bmatrix}
25 & -22 & 23 & -19 & 15 & -25 \\
35 & -37 & 36 & -26 & 16 & -17 \\
42 & -43 & 35 & -16 & 1 & 20 \\
47 & -45 & 29 & -6 & -10 & 28 \\
52 & -45 & 21 & 6 & -16 & 39 \\
55 & -44 & 12 & 15 & -21 & 12 \\
58 & -40 & 2 & 21 & -17 & 16 \\
61 & -36 & -6 & 24 & -14 & -22 \\
63 & -32 & -16 & 27 & -6 & -17 \\
65 & -25 & -22 & 25 & 1 & -39 \\
66 & -20 & -28 & 19 & 8 & -28 \\
67 & -13 & -32 & 15 & 15 & -30 \\
67 & -7 & -36 & 8 & 19 & -13 \\
67 & 0 & -36 & 0 & 18 & 0 \\
67 & 7 & -36 & -8 & 19 & 13 \\
67 & 13 & -32 & -15 & 15 & 30 \\
66 & 20 & -28 & -19 & 8 & 28 \\
65 & 25 & -22 & -25 & 1 & 39 \\
63 & 32 & -16 & -27 & -6 & 17 \\
61 & 36 & -6 & -24 & -14 & 22 \\
58 & 40 & 2 & -21 & -17 & -16 \\
55 & 44 & 12 & -15 & -21 & -12 \\
52 & 45 & 21 & -6 & -16 & -39 \\
47 & 45 & 29 & 6 & -10 & -28 \\
42 & 43 & 35 & 16 & 1 & -20 \\
35 & 37 & 36 & 26 & 16 & 17 \\
25 & 22 & 23 & 19 & 15 & 25
\end{bmatrix}
$$

$$\begin{array}{c} \text{(0,0)5} \qquad\qquad \text{31 Points} \end{array}$$

$$(MU)^{-1} = \begin{bmatrix} \frac{1}{31} & 0 & \frac{-1}{341} & 0 & \frac{69493}{21085053} & 0 \\ 0 & \frac{1}{2480} & 0 & \frac{-719}{596280} & 0 & \frac{16792999}{15711978000} \\ 0 & 0 & \frac{1}{27280} & 0 & \frac{-698837}{5060412720} & 0 \\ 0 & 0 & 0 & \frac{1}{119256} & 0 & \frac{-263317}{12569582400} \\ 0 & 0 & 0 & 0 & \frac{1}{1483992} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{12648000} \end{bmatrix}$$

$$\begin{array}{c} \text{(0,0)5} \qquad\qquad \text{31 Points} \end{array}$$

$$M^* = \begin{bmatrix} 1 & -15 & 25 & -15 & 24 & -25 \\ 1 & -14 & 20 & -9 & 8 & 0 \\ 1 & -13 & 15 & -4 & -3 & 13 \\ 1 & -12 & 12 & 0 & -10 & 17 \\ 1 & -11 & 6 & 3 & -14 & 16 \\ 1 & -10 & 4 & 5 & -12 & 11 \\ 1 & -9 & 0 & 8 & -17 & 5 \\ 1 & -8 & -3 & 7 & -7 & -3 \\ 1 & -7 & -5 & 8 & -9 & -8 \\ 1 & -6 & -7 & 9 & -2 & -8 \\ 1 & -5 & -10 & 6 & -1 & -19 \\ 1 & -4 & -11 & 8 & 8 & -10 \\ 1 & -3 & -13 & 3 & 6 & -16 \\ 1 & -2 & -12 & 5 & 11 & -5 \\ 1 & -1 & -14 & 1 & 10 & -8 \\ 1 & 0 & -14 & 0 & 16 & 0 \\ 1 & 1 & -14 & -1 & 10 & 8 \\ 1 & 2 & -12 & -5 & 11 & 5 \\ 1 & 3 & -13 & -3 & 6 & 16 \\ 1 & 4 & -11 & -8 & 8 & 10 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 15 & 25 & 15 & 24 & 25 \end{bmatrix}$$

(-1/2, -1/2)5 31 Points

$$
(MU)^{-1} = \begin{bmatrix}
\frac{1}{1049} & 0 & \frac{-54491}{6329666} & 0 & \frac{1013514917}{79643022445} & 0 \\
0 & \frac{1}{1734} & 0 & \frac{-51043}{26603028} & 0 & \frac{9919623359}{3730010555880} \\
0 & 0 & \frac{1}{12\text{[illegible]}68} & 0 & \frac{-717053}{1822147320} & 0 \\
0 & 0 & 0 & \frac{1}{92052} & 0 & \frac{-455551}{10325288736} \\
0 & 0 & 0 & 0 & \frac{1}{603960} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6730080}
\end{bmatrix}
$$

(-1/2, -1/2)5 31 Points

$$
M^* = \begin{bmatrix}
72 & -25 & 24 & -23 & 18 & -23 \\
52 & -10 & 4 & 2 & -7 & 17 \\
43 & -7 & 3 & 1 & -3 & 5 \\
38 & -6 & 1 & 2 & -4 & 9 \\
34 & -5 & 0 & 6 & -8 & 4 \\
32 & -3 & 0 & 3 & -1 & 6 \\
30 & -4 & -3 & 5 & -5 & -4 \\
29 & -3 & -2 & 5 & -3 & 1 \\
28 & -2 & -2 & 5 & 0 & -9 \\
27 & -2 & -4 & 4 & -1 & 1 \\
26 & -2 & -3 & 3 & 2 & -10 \\
26 & -1 & -3 & 4 & 1 & -3 \\
25 & -1 & -5 & 2 & 3 & -5 \\
25 & 0 & -4 & 0 & 2 & -4 \\
25 & -1 & -4 & 3 & 5 & 0 \\
25 & 0 & -4 & 0 & 2 & 0 \\
25 & 1 & -4 & -3 & 5 & 0 \\
25 & 0 & -4 & 0 & 2 & 4 \\
25 & 1 & -5 & -2 & 3 & 5 \\
26 & 1 & -3 & -4 & 1 & 3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
72 & 25 & 24 & 23 & 18 & 23
\end{bmatrix}
$$

(1/2, 1/2)5 31 Points

$$
(MU)^{-1} = \begin{bmatrix}
\frac{1}{1504} & 0 & \frac{-11833}{10877680} & 0 & \frac{97679037}{77902544885} & 0 \\
0 & \frac{1}{7878} & 0 & \frac{-157891}{502206744} & 0 & \frac{18611261727}{3671549804260} \\
0 & 0 & \frac{1}{57860} & 0 & \frac{-2601659}{39780022920} & 0 \\
0 & 0 & 0 & \frac{1}{382488} & 0 & \frac{-762265}{6711134448} \\
0 & 0 & 0 & 0 & \frac{1}{2750088} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2105520}
\end{bmatrix}
$$

(1/2, 1/2)5 31 Points

$$
M^* = \begin{bmatrix}
21 & -19 & 24 & -24 & 24 & -2 \\
29 & -32 & 39 & -35 & 29 & -4 \\
35 & -38 & 39 & -25 & 8 & 5 \\
40 & -41 & 35 & -14 & -9 & -2 \\
44 & -42 & 29 & -1 & -19 & 6 \\
47 & -41 & 20 & 11 & -33 & 2 \\
50 & -39 & 12 & 22 & -31 & 3 \\
52 & -37 & 1 & 27 & -28 & -2 \\
54 & -34 & -6 & 33 & -22 & 1 \\
56 & -30 & -14 & 34 & -16 & -3 \\
57 & -25 & -22 & 32 & 0 & -3 \\
58 & -21 & -29 & 30 & 9 & -3 \\
59 & -16 & -32 & 24 & 14 & -1 \\
60 & -10 & -38 & 18 & 29 & -7 \\
60 & -6 & -38 & 7 & 30 & 3 \\
60 & 0 & -40 & 0 & 30 & 0 \\
60 & 6 & -38 & 7 & 30 & -3 \\
60 & 10 & -38 & -18 & 29 & 7 \\
59 & 16 & -32 & -24 & 14 & 1 \\
58 & 21 & -29 & -30 & 9 & 3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
21 & 19 & 24 & 24 & 24 & 2
\end{bmatrix}
$$

(0,0)5 35 Points

$$(MU)^{-1} = \begin{bmatrix} \frac{1}{35} & 0 & \frac{-1}{259} & 0 & \frac{430221}{130187645} & 0 \\ 0 & \frac{1}{3570} & 0 & \frac{-917}{1075500} & 0 & \frac{172928117}{195464327625} \\ 0 & 0 & \frac{1}{26418} & 0 & \frac{-1151687}{10623311832} & 0 \\ 0 & 0 & 0 & \frac{1}{215100} & 0 & \frac{-1212913}{89355121200} \\ 0 & 0 & 0 & 0 & \frac{1}{2412744} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{24924720} \end{bmatrix}$$

(0,0)5 35 Points

$$M^* = \begin{bmatrix} 1 & -17 & 17 & -17 & 22 & -25 \\ 1 & -16 & 14 & -11 & 9 & -3 \\ 1 & -15 & 11 & -6 & 0 & 10 \\ 1 & -14 & 9 & -1 & -8 & 15 \\ 1 & -13 & 6 & 0 & -8 & 17 \\ 1 & -12 & 3 & 7 & -15 & 11 \\ 1 & -11 & 3 & 4 & -12 & 14 \\ 1 & -10 & -1 & 10 & -11 & -5 \\ 1 & -9 & -2 & 7 & -7 & 9 \\ 1 & -8 & -3 & 11 & -11 & -19 \\ 1 & -7 & -5 & 7 & 0 & 1 \\ 1 & -6 & -6 & 9 & -1 & -21 \\ 1 & -5 & -7 & 7 & 3 & -11 \\ 1 & -4 & -8 & 8 & 4 & -9 \\ 1 & -3 & -8 & 4 & 11 & -15 \\ 1 & -2 & -9 & 3 & 7 & -9 \\ 1 & -1 & -10 & 2 & 11 & 0 \\ 1 & 0 & -8 & 0 & 12 & 0 \\ 1 & 1 & -10 & -2 & 11 & 0 \\ 1 & 2 & -9 & -3 & 7 & 9 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 17 & 17 & 17 & 22 & 25 \end{bmatrix}$$

(-1/2, -1/2)5 35 Points

$$
(MU)^{-1} = \begin{bmatrix}
\frac{1}{1197} & 0 & \frac{-8857}{989254} & 0 & \frac{4662798865}{375638539626} & 0 \\
0 & \frac{1}{1812} & 0 & \frac{-34181}{16351488} & 0 & \frac{3310128593}{1425004926720} \\
0 & 0 & \frac{1}{14876} & 0 & \frac{-1130045}{3765799896} & 0 \\
0 & 0 & 0 & \frac{1}{108288} & 0 & \frac{-57065}{1887423744} \\
0 & 0 & 0 & 0 & \frac{1}{1012584} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{12549360}
\end{bmatrix}
$$

(-1/2, -1/2)5 35 Points

$$
M^* = \begin{bmatrix}
76 & -22 & 22 & -18 & 18 & -23 \\
55 & -9 & 4 & 1 & -6 & 15 \\
45 & -6 & 3 & 0 & -2 & 4 \\
40 & -5 & 2 & 2 & -4 & 8 \\
36 & -4 & 0 & 2 & -6 & 7 \\
34 & -4 & 0 & 3 & -4 & 4 \\
32 & -3 & -1 & 5 & -3 & 0 \\
30 & -3 & -2 & 1 & -3 & 3 \\
29 & -2 & -2 & 6 & -2 & -8 \\
28 & -2 & -2 & 1 & -4 & 2 \\
27 & -1 & -2 & 5 & 4 & -7 \\
27 & -2 & -3 & 2 & -3 & -5 \\
26 & -1 & -4 & 2 & 5 & -4 \\
26 & -1 & -3 & 2 & -2 & -4 \\
25 & -1 & -3 & 2 & 5 & -4 \\
25 & -0 & -4 & 1 & 3 & -4 \\
25 & -0 & -3 & 0 & 1 & 1 \\
25 & 0 & -4 & 0 & 6 & 0 \\
25 & +0 & -3 & 0 & 1 & -1 \\
25 & +0 & -4 & -1 & 3 & 4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
76 & 22 & 22 & 18 & 18 & 23
\end{bmatrix}
$$

(1/2, 1/2)5 35 Points

$$
(MU)^{-1} = \begin{bmatrix}
\frac{1}{2142} & 0 & \frac{-42776}{35057043} & 0 & \frac{966527411}{591020501400} & 0 \\
0 & \frac{1}{13392} & 0 & \frac{-171083}{569387664} & 0 & \frac{103166494081}{27101429337240} \\
0 & 0 & \frac{1}{65466} & 0 & \frac{-2502859}{37525111200} & 0 \\
0 & 0 & 0 & \frac{1}{510204} & 0 & \frac{-2594933}{38855095824} \\
0 & 0 & 0 & 0 & \frac{1}{3439200} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4569360}
\end{bmatrix}
$$

(1/2, 1/2)5 35 Points

$$
M^{*} = \begin{bmatrix}
25 & -24 & 18 & -19 & 16 & -2 \\
35 & -41 & 30 & -29 & 21 & -5 \\
42 & -49 & 31 & -23 & 9 & 6 \\
48 & -53 & 29 & -15 & -1 & -6 \\
53 & -56 & 26 & -7 & -12 & 11 \\
57 & -55 & 19 & 4 & -15 & -2 \\
60 & -55 & 15 & 12 & -22 & 6 \\
63 & -52 & 7 & 17 & -22 & 0 \\
66 & -49 & 2 & 24 & -19 & -1 \\
68 & -45 & -4 & 25 & -16 & 2 \\
70 & -41 & -11 & 29 & -10 & -2 \\
72 & -36 & -15 & 27 & -4 & -7 \\
73 & -30 & -20 & 26 & 2 & 0 \\
74 & -25 & -24 & 22 & 9 & 0 \\
75 & -19 & -28 & 17 & 15 & -12 \\
76 & -12 & -28 & 14 & 17 & 7 \\
76 & -7 & -32 & 5 & 21 & -6 \\
76 & 0 & -30 & 0 & 22 & 0 \\
76 & 7 & -32 & -5 & 21 & 6 \\
76 & 12 & -28 & -14 & 17 & -7 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
25 & 24 & 18 & 19 & 16 & 2
\end{bmatrix}
$$

APPENDIX C

COMPUTATION OF THE ORTHOGONAL POLYNOMIALS AND WEIGHTING MATRIX FOR GIVEN MULTIPLIERS

In reference (2) a method for going from the weighting matrix and the corresponding orthogonal polynomials to the w's and m's was indicated. Since rounding of the w's occurs in the construction of a convenient set of m's it is the purpose of this appendix to show a method of construction of the orthogonal polynomials and the weighting matrix from the m's.

Suppose there are r + 1 discrete points,

$\vec{u}^n$ = the vector whose components are the n^{th} powers of the grid point arguments,

$\vec{m}_k$ = the vector whose components are the multipliers for finding the mean k^{th} difference,

$\vec{Q}_k$ = the vector whose components are evaluations of the k^{th} orthogonal polynomial for the grid point arguments,

μ^{-1} = the $(r + 1) \times (r + 1)$ weighting matrix,

To construct $\vec{Q}_k$ find least squares (k - 1)st degree fit, using known m's, to

$$u^k = A_0 Q_0 + A_1 Q_1 + \dots + A_{k-1} Q_{k-1} + A_k Q_k.$$

If P_{k-1} is this (k-1)st degree polynomial fit

$$\vec{u}^k - \vec{P}_{k-1} = A_k \vec{Q}_k$$

Take all components of $\vec{Q}_o$ to be 1, construct $\vec{Q}_1, \vec{Q}_2, \ldots, \vec{Q}_r$ and let

$$Q = \left[\vec{Q}_o \; \vec{Q}_1 \; \vec{Q}_2 \; \ldots \; \vec{Q}_r \right] \quad ,$$

$$M^* = \left[\vec{m}_o \; \vec{m}_1 \; \vec{m}_2 \; \ldots \; \vec{m}_r \right] \quad .$$

Q and M* are (r + 1) x (r + 1) nonsingular matrices. From reference (2).

$$M^* = \mu^{-1}Q, \text{ and therefore}$$

$$M^*Q^{-1} = \mu^{-1}, \text{ the required weighting matrix.}$$

Note that the m's obtained from rounded w's give a true least squares fit with respect to a "rounded" weighting matrix.

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13. ABSTRACT

The problem of developing orthogonal polynomials for finding a least-squares best approximation to a continuous function over a given interval is solved by a method of finding the mean k^{th} derivative of the function over that interval. A parallel method, using mean differences, is developed for fitting discrete data.